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# Effect of Correlation on Reliability-Based Design of Composite Plate for Buckling

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**Effects of correlation between random variables on reliability and reliability-based design of a composite plate subject to buckling are investigated. The reliability is evaluated by modeling buckling failure as a series system consisting of potential eigenmodes. The mode reliability is evaluated by the first-order reliability method, where ply orientation angles of individual layers and applied loads are treated as correlated random variables. Then the system reliability is approximated by Ditlevsen's upper bound. The reliability is maximized in terms of the mean ply orientation angles. Through numerical calculations, it is clarified that the reliability-based design changes significantly in terms of the correlation coefficients. It is also shown that the reliability-based design ignoring correlation is sometimes less safe than even a deterministic buckling load maximization design when the random variables are correlated. This indicates that the estimation of correlation is very important for the reliability analysis and design of the composite plate.**

## Introduction

LAMINATED plates are widely used in structural applications because of their high specific strength and stiffness. For any plate, the presence of in-plane loads may cause buckling. Therefore, many studies have been conducted on the buckling load maximization design of composite plates.[1]

However, most studies yield the optimum fiber orientation angles under deterministic conditions, where the material properties and the load conditions are assumed to have no variations. It has been known that such a deterministic optimum design is strongly anisotropic and sensitive to change in loading conditions.[1,2] Therefore, it is necessary to consider the effect of such variations by applying the structural reliability theory.[3] The reliability-based design that maximizes the structural reliability is especially important.[3–5]

Reliability analysis and reliability-based design under the in-plane strength using the first ply failure criterion have already been studied.[6–8] The studies have shown that reliability increases as the number of fiber axes is increased and that the reliability-based design approaches a quasi-isotropic configuration. This design is very different from the deterministic optimum design in which the orientation angle runs along the loading direction.

The authors have presented the reliability analysis[9] and the reliability-based design[10] of a laminated plate subject to buckling when the material constants, the ply orientation angles, and the applied loads have random variations. It was shown that variations of Young's modulus in the fiber direction $E_1$, the ply orientation angle $\theta$, and the applied loads have dominant effects on the reliability. Comparing the reliability-based design with the deterministic buckling maximization design, the following conclusions are remarked[10]: The buckling load will reach the maximum when the buckling modes are repeated. On the other hand, the reliability will reach the maximum when the mode reliabilities of the critical failure modes are well balanced. Hence, the reliability-based design is very different from deterministic design.

However, the preceding discussion is limited to the case where the random variables are independent of each other. In actual situations, correlation exists between some random variables. For example, the applied loads are often correlated. Correlation between the ply orientation angles will arise from the manufacturing process.

It is known that generally correlation greatly affects reliability.[3] However, few studies exist to quantify the effect of correlation between random variables on the reliability analysis and design of actual structures. Cederbaum and Arbocz[11] studied the effect of an initial imperfection on the reliability of a long isotropic cylindrical shell subject to buckling. The initial imperfection is described by a two-dimensional Fourier series with random amplitude. The study shows that a higher correlation between the two Fourier parameters reduces the reliability. Frangopol et al.[12] studied the effects of the load path and the load correlation on the reliability of reinforced concrete columns subject to an axial compression load and a bending moment. It was shown that the high value of correlation results in the higher reliability in most of the compression failure region. On the other hand, in the tensile failure region, the high value of correlation results in the lower reliability.

In this paper, correlation between the ply orientation angles or the applied loads is investigated to determine its effect on the reliability analysis and design of a composite plate subject to buckling.

## Buckling Analysis

In practical structural applications, a laminated plate is mainly stacked symmetrically with respect to the midplane to avoid bending–extension coupling. However, the actual stacking sequence will not be symmetric due to the random variations of material properties and orientation angle of each ply, even though the mean stacking sequence is assumed to be symmetric. Consequently, buckling analysis for an asymmetric laminate is required.

### Constitutive Equations

For a laminated plate, as shown in Fig. 1, the constitutive equations are given as follows[13]:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} \tag{1}$$

The $\boldsymbol{A}$ and $\boldsymbol{D}$ matrices are extensional and flexural stiffness matrices, respectively. The $\boldsymbol{A}$ matrix relates the in-plane stress resultants $\boldsymbol{N}$ to the midplane strains $\varepsilon^0$, whereas the $\boldsymbol{D}$ matrix relates the moment resultants $\boldsymbol{M}$ to the curvature $\boldsymbol{\kappa}$. The $\boldsymbol{B}$ matrix, on the other hand, relates the in-plane stress resultants to the curvature and the moment resultants to the midplane strains, and hence, it is called the bending–extension coupling matrix.

### Reduced Bending Stiffness Method

Because the asymmetric laminate has bending–extension coupling, it is difficult to directly solve the governing equation. Therefore, a reduced bending stiffness method is used here as an

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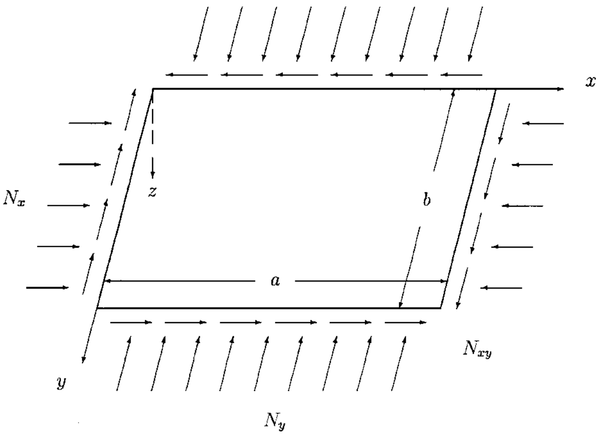



**Fig. 1 Laminated plate geometry and applied loadings.**

approximation method.[14] The method reduces the problem to an equivalent anisotropic bending problem without bending–extension coupling. Then Galerkin's method for the analysis of a symmetric laminate plate can be directly utilized.[13]

Multiplying the first equation of the constitutive equation (1) by $A^{-1}$ and substituting the result into the second yields the following equation:

$$\begin{Bmatrix} \varepsilon^0 \\ M \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ (-B^*)^T & D^* \end{bmatrix} \begin{Bmatrix} N \\ \kappa \end{Bmatrix} \tag{2}$$

where

$$A^* = A^{-1}, \qquad B^* = -A^{-1}B, \qquad D^* = D - BA^{-1}B \tag{3}$$

By using the constitutive relation of the plate in the form of Eq. (2), the strain energy $U$ of a laminated rectangular plate can be expressed in the following form:

$$\begin{aligned} U &= \frac{1}{2}\int_0^b\int_0^a \left(\varepsilon^{0^T} N + \kappa^T M\right) \mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^b\int_0^a (N^T A^* N + \kappa^T D^* \kappa)\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{4}$$

where $a$ and $b$ are the plate dimensions of the $x$ and $y$ directions, respectively. There is no bending–extension coupling appearing in this representation. This suggests that the asymmetric laminated plate problem can be solved approximately in the following uncoupled form:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D^* \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} \tag{5}$$

Thus, the governing equation of a symmetric laminated plate can be utilized by substituting $D^*_{ij}$ for the bending stiffness $D_{ij}$.

#### Galerkin's Method

Consider a simply supported rectangular laminated plate with dimensions $a$ and $b$ subjected to uniform biaxial compression and shear loads ($N_x$, $N_y$, $N_{xy}$), as shown in Fig. 1. The out-of-plane displacement function satisfying the boundary condition is given as follows:

$$w = \sum_{m=1}^{M}\sum_{n=1}^{N} \phi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6}$$

The following set of algebraic equations is obtained from Galerkin's method[13]:

$$\begin{aligned} &\pi^4\left[D^*_{11}m^4 + 2\left(D^*_{12} + 2D^*_{66}\right)m^2n^2R^2 + D^*_{22}n^4R^4 - N_x\frac{m^2a^2}{\pi^2} \right. \\ &\left. - N_y\frac{n^2R^2a^2}{\pi^2}\right]\phi_{mn} - 32\pi^2 mnR\sum_{i=1}^{M}\sum_{j=1}^{N} M_{ij} \\ &\times\left[(m^2+i^2)D^*_{16} + (n^2+j^2)D^*_{26}R^2 + N_{xy}\frac{a^2}{\pi^2}\right]\phi_{ij} = 0 \\ &(m = 1,\ldots,M;\, n = 1,\ldots,N) \end{aligned} \tag{7}$$

where $R$ is the aspect ratio $a/b$ and

$$M_{ij} = \begin{cases} \dfrac{ij}{(m^2-i^2)(n^2-j^2)} & (m+i = \text{odd} \quad \text{and} \quad n+j = \text{odd}) \\ 0 & (\text{otherwise}) \end{cases} \tag{8}$$

Equation (7) yields a set of $M \times N$ homogeneous equations. This set of equations can be reduced to an eigenvalue equation for $\phi_{mn}$ when the stiffness terms and the loading terms are separated. That is,

$$[K - \lambda K_G]\phi = 0 \tag{9}$$

where $\phi$ is a vector consisting of $\phi_{mn}$, $K$ is the stiffness matrix, and $K_G$ is the geometry stiffness matrix.

The resulting eigenvalue equation (9) can be separated into two sets, one for $m+n = $ even and the other for $m+n = $ odd. The minimum eigenvalue $\lambda_{\min}$ of the two problems corresponds to the buckling load factor under the applied load ($N_x$, $N_y$, $N_{xy}$). That is, the plate will buckle when the load reaches $\lambda_{\min} \times (N_x, N_y, N_{xy})$.

## Reliability Analysis

Buckling failure occurs when the minimum eigenvalue is less than unity. In other words, all of the eigenvalues should be larger than unity so as not to buckle the plate. Therefore, the buckling failure event can be modeled as a series system consisting of eigenmodes.

The limit state functions of the series system are written as follows:

$$g_i(u) = \lambda_i(u) - 1 \geq 0 \qquad (i = 1,\ldots,N_f) \tag{10}$$

where $\lambda_i$ is the eigenvalue corresponding to the $i$th failure mode, $N_f$ is the number of failure modes, and $u$ is the standardized, uncorrelated, and normal vector, which is transformed from the basic random vector $x$, i.e., $u = T(x)$. The ply orientation angles of individual layers or of the applied loads are treated as random variables.

There exist $M \times N$ failure modes, and all of the eigenmodes may have some probabilities of occurrence. However, some have high probabilities, and the others have relatively low ones. It is presumed that an eigenmode corresponding to a lower eigenvalue will be more critical than that of a higher eigenvalue. Because the lower mode is always closer to the failure surface than the higher mode, $N_f$ eigenmodes taken from the minimum eigenvalue to the $N_f$th in an ascending order at the mean value point are selected as dominant failure modes.

#### Transformation into Standard Normal Distribution Space

As stated earlier, random variables $X = (X_1,\ldots,X_n)^T$ should be transformed into an independent standard normal vector $U = (U_1,\ldots,U_n)^T$, i.e., $U = T^{-1}(X)$ using the first-order reliability method (FORM).[3]

Consider the case where the random variables have dependent normal distribution. The covariance matrix is defined as follows:

$$C_X = \begin{bmatrix} \sigma^2_{X_1} & \rho_{X_1X_2}\sigma_{X_1}\sigma_{X_2} & \cdots & \rho_{X_1X_n}\sigma_{X_1}\sigma_{X_n} \\ & \sigma^2_{X_2} & \cdots & \rho_{X_2X_n}\sigma_{X_2}\sigma_{X_n} \\ & & \ddots & \vdots \\ & \text{symmetry} & & \sigma^2_{X_n} \end{bmatrix} \tag{11}$$

where $\sigma_{X_i}$ is a standard deviation of $X_i$ and $\rho_{X_iX_j}$ is a correlation coefficient between $X_i$ and $X_j$. Using the orthogonal transformation, $X$ is transformed into random variables $Y$ without any correlation:

$$Y = A^T X \tag{12}$$

where $A$ is a transformation matrix whose row consists of an orthonormal eigenvector of $C_X$. Using this transformation, the covariance matrix of $Y$ becomes a diagonal matrix

$$C_Y = A^T C_X A = \mathrm{diag}\left(\sigma^2_{Y_1}, \sigma^2_{Y_2}, \ldots, \sigma^2_{Y_n}\right) \tag{13}$$

where $\sigma_{Y_i}$ is a standard deviation of $Y_i$. The variance of the new variable $\sigma_{Y_i}^2$ is equal to an eigenvalue of $\boldsymbol{C}_X$, and the mean of the variables $\boldsymbol{\mu}_Y$ is calculated from Eq. (12) as $\boldsymbol{\mu}_Y = \boldsymbol{A}^T\boldsymbol{\mu}_X$.

Then $\boldsymbol{Y}$ is transformed into the standard normal distribution

$$\boldsymbol{U} = \boldsymbol{C}_Y^{-\frac{1}{2}}(\boldsymbol{Y} - \boldsymbol{\mu}_Y) = \left(\boldsymbol{A}^T\boldsymbol{C}_X\boldsymbol{A}\right)^{-\frac{1}{2}}\boldsymbol{A}^T(\boldsymbol{X} - \boldsymbol{\mu}_X) \tag{14}$$

where $\boldsymbol{C}_Y^{-1/2}$ is a diagonal matrix whose diagonal elements consist of $1/\sigma_{Y_i}$.

Note that there exist lower limits of the correlation coefficients $\rho_{X_i,X_j}$. Because of physical constraints, the standard deviation $\sigma_{Y_i}$ of the new variable must be a real number. That is, the original covariance matrix $\boldsymbol{C}_X$ should be positive definite. Therefore, the lower limit of the negative value of the correlation coefficient is bounded by the positive definite condition of the covariance matrix. For example, consider the case where all of the random variables take the same standard deviation $\sigma$ and all of the correlation coefficients take the same value $\rho$ between any random variables of the $n$-dimensional problem. Because the covariance matrix consists of the diagonal elements $\sigma^2$ and the off-diagonal elements $\rho\sigma^2$, the positive definite condition is written as follows:

$$(-1)^{n-1}(\rho - 1)^{n-1}\{1 + (n-1)\rho\} \geq 0 \tag{15}$$

Therefore, the bounds can be expressed as follows:

$$-1/(n-1) \leq \rho \leq 1 \tag{16}$$

**Mode Reliability**

The mode reliability is evaluated for each eigenmode using FORM. Because the eigenvalue is a nonlinear function of the random variables such as ply orientation angles or the applied loads, the reliability searching process of the $i$th failure mode is formulated as the following nonlinear programming problem[3]:

Minimize

$$\beta_i = \sqrt{\boldsymbol{u}^T\boldsymbol{u}}$$
$$\text{subject to} \quad g_i(\boldsymbol{u}) = \lambda_i(\boldsymbol{u}) - 1 = 0 \qquad (i = 1, \ldots, N_f) \tag{17}$$

The solution gives the mode reliability index, and the obtained point $\boldsymbol{u}^*$ is called a design point, in which the limit state function $g_i(\boldsymbol{u})$ is linearized:

$$g_i \approx -\boldsymbol{\alpha}_i^T\boldsymbol{u} + \beta_i \tag{18}$$

where $\boldsymbol{\alpha}$ is the unit vector of the negative direction cosine of $\boldsymbol{u}^*$:

$$\boldsymbol{\alpha}_i = \boldsymbol{u}^*\big/\beta_i = -\nabla_u g_i(\boldsymbol{u}^*)\big/\left|\nabla_u g_i(\boldsymbol{u}^*)\right| \tag{19}$$

where $\nabla_u g_i(\boldsymbol{u}^*)$ is the gradient of the limit state function $g_i$ at the design point $\boldsymbol{u}^*$.

Then the mode failure probability is evaluated by using the standardized normal distribution function

$$P_i = \Phi(-\beta_i) = \int_{-\infty}^{-\beta_i} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t \tag{20}$$

**Mode Tracking Strategy**

To evaluate the limit state function, the corresponding eigenvalue should be evaluated by the buckling analysis. In the analysis, the eigenvalues and eigenvectors are ordered by eigenvalue magnitude, and the specified mode must be referenced by a number. When the random vector is moved in the iteration process of Eq. (17), mode crossing may occur. If these crossings are not tracked, the reliability may be evaluated by using modes different from the intended one. Therefore, a mode tracking strategy is utilized to keep track of the intended mode.

The strategy has been developed for the experimental modal identification[15] and the structural dynamics optimization problem.[16,17] In this study, the cross orthogonality check method based on an eigenvector orthogonality is utilized. At each iteration in the $\beta$-searching process, the following eigenvector orthogonality check is performed:

$$C_{ij} = \boldsymbol{\phi}_i^{(k-1)^T}\boldsymbol{K}_G^{(k)}\boldsymbol{\phi}_j^{(k)} \tag{21}$$

where superscripts $(k)$ and $(k-1)$ denote the current and the previous iterations, respectively. If the $i$th mode at the previous iteration corresponds to the current $j$th mode, $C_{ij}$ is nearly equal to unity. Otherwise, $C_{ij}$ is nearly equal to zero. Using this strategy, the corresponding mode number can always be monitored.

When the eigenvector change is large due to the large design change, the cross check may fail. In this study, the strategy is modified to improve the accuracy. Replacing the previous eigenvector by the first-order Taylor expansion around the position previous to the current position, the accuracy will be improved. However, the derivatives of the eigenvectors require high computational costs. The $\beta$-searching process is performed around the mean value point ($\boldsymbol{u} = \boldsymbol{0}$) for almost all cases. Moreover, the design change in the reliability analysis is relatively smaller than that in the structural optimization problem. Therefore, the cross check is performed between the current eigenvector and the Taylor expanded vector around the mean eigenvector to the current position. Thus,

$$C_{ij} = \left(\boldsymbol{\phi}_i^{(0)} + \nabla\boldsymbol{\phi}_i^{(0)^T}\boldsymbol{u}\right)^T\boldsymbol{K}_G^{(k)}\boldsymbol{\phi}_j^{(k)} \tag{22}$$

where $\boldsymbol{\phi}_i^{(0)}$ and $\nabla\boldsymbol{\phi}_i^{(0)}$ are the mean eigenvector and the eigenvector derivative with respect to the random vector at the mean value point, respectively.

**Stacking Sequence and System Reliability**

The stacking sequence corresponding to the design point often will be asymmetric. Even if the sequence is turned upside down, the plate has the same structural property. However, the corresponding point in the $\boldsymbol{U}$ space is different from the design point. But the point has the same distance from the origin as the design point because the mean laminate configuration is set to be symmetric. Therefore, the point should be taken as another design point. That is, each failure mode has two design points. Totally, $2 \times N_f$ design points should be considered to evaluate the system reliability.

The system failure probability $P_f$ is approximated by using Ditlevsen's bounds[18]:

$$P_L \leq P_f \leq P_U, \qquad P_L = P_1 + \sum_{i=2}^{2N_f}\max\left(P_i - \sum_{j=1}^{i-1}P_{ij}, 0\right)$$
$$P_U = \sum_{i=1}^{2N_f}P_i - \sum_{i=2}^{2N_f}\max_{j<i}P_{ij} \tag{23}$$

where $P_i$ is the failure probability of the mode $i$ and $P_{ij}$ is the joint failure probability of the modes $i$ and $j$.

The system reliability index $\beta$ is estimated by the reliability indices of the lower and the upper bounds, which are estimated by using the inverse of the standardized normal distribution function:

$$\beta_U \leq \beta \leq \beta_L, \qquad \beta_L = -\Phi^{-1}(P_L), \qquad \beta_U = -\Phi^{-1}(P_U) \tag{24}$$

## Reliability-Based Design

The problem is to find the laminate configuration such that the system reliability index may be maximized in terms of the mean ply orientation angle of each ply. The total number of the plies is set to be constant. The system reliability of the series system is approximated by Ditlevsen's[18] upper bound. Therefore, $\beta_U$ is set as the objective function. The optimization problem is formulated as follows:

Maximize

$$\beta_U(\boldsymbol{u}, \boldsymbol{t})$$
$$\text{subject to} \quad g_i(\boldsymbol{u} = \boldsymbol{0}, \boldsymbol{t}) \geq 0 \qquad (i = 1, \ldots, N_f) \tag{25}$$

where $\boldsymbol{t}$ is the design vector of ply orientation angles and $\boldsymbol{u}$ is the random vector in the reliability analysis. The inequality constraints mean that the origin of the $\boldsymbol{U}$ space in the reliability analysis of the current design $\boldsymbol{t}$ should lie in the safety region.

The reliability-based optimization problem becomes a nested problem with two levels of optimization, i.e., the main design problem and the reliability analysis.

In this optimization algorithm, the gradient of the reliability index is required to obtain the search direction. The derivative of the system reliability (upper bound $\beta_U$) with respect to the design variable is evaluated by the method developed by Sørensen[4] (also see Ref. 5).

## Numerical Calculations

The reliability analysis is performed for the deterministic buckling load maximization design, which is then compared with reliability-based optimum designs. Numerical examples are given for a simply supported tetra-axial angle-ply laminated plate $[+\theta_1/-\theta_1/-\theta_2/+\theta_2]_s$ of aspect ratio $R = 2.0$ made of graphite/epoxy (T300/5208) under several load conditions of biaxial compression. The material constants are $E_1 = 181.0$ GPa, $E_2 = 10.7$ GPa, $\nu_1 = 0.28$, and $E_6 = 7.17$ GPa.

Correlation between the following two types of variables is considered: 1) the applied loads and 2) the ply orientation angles between layers. These two types of variables have dominant effects on the reliability when all of the variables follow independent normal distribution.[9] The assigned variables are assumed here to be normally distributed, whereas the remaining parameters are deterministic. The mean applied load for case 1 or the deterministic applied load for case 2 is set to 0.8 times the maximum buckling load of the angle-ply plate (Table 1). In the reliability-based optimization, the orientation angles $(\theta_1, \theta_2)$ for case 1 or the mean orientation angles $(\bar{\theta}_1, \bar{\theta}_2)$ for case 2 are treated as design variables. Finally, the accuracy of the reliability-based optimization method is confirmed by a Monte Carlo simulation.

### Deterministic Design

The deterministic optimum designs that maximize the buckling load in terms of ply orientation angles $(\theta_1, \theta_2)$ are listed in Table 1 for the two load cases of $N_y/N_x = 0.25$ and 0.50. The applied load is standardized for the plate dimensions as follows[19,20]:

$$N^* = \frac{12a^2}{\pi^2 h^3 R^2} N \tag{26}$$

where $h$ is a plate thickness. Because the standardized applied load is set to $N_x^* = 1$ GPa, the plate will be buckled when the load reaches the minimum eigenvalue $\lambda_{\min}$ (GPa). Under these load conditions, the maximum buckling load is obtained as the repeated eigenvalue. Concerning the optimum laminate configuration, the orientation angle of the surface layer is larger than that of the midplane.

### Correlation Between Applied Loads

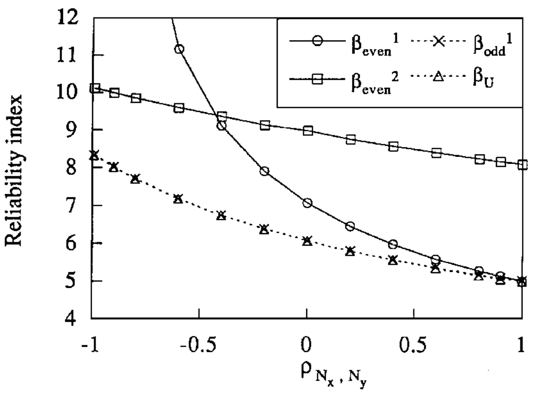
Consider the case where applied loads are correlated variables, whereas the material constants and the ply orientation angles are set to be deterministic. The variation of the shear force $N_{xy}$ is shown to have little effect on the reliability when all of the variables are independent.[9] Therefore, the shear force is set to be constant, and the axial compression loads $N_x$ and $N_y$ are assumed to be the correlated random variables with coefficients of variation (COV) of the applied loads, $\mathrm{COV}(N_x) = \mathrm{COV}(N_y) = 0.05$ and $-1 \le \rho_{N_x,N_y} \le 1$.

*Reliability Analysis*

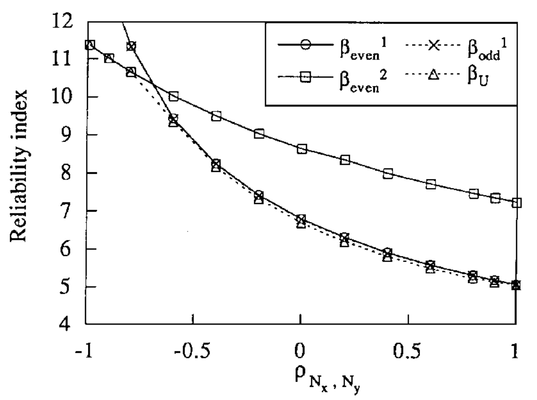
Reliability analyses are performed on the deterministic designs under load cases of $\bar{N}_y/\bar{N}_x = 0.25$ and 0.5. The changes of mode reliabilities in terms of the correlation coefficients are shown in Fig. 2. As correlation increases, the reliability is decreased. This is because the variation of the resultant load direction $N_y/N_x$ is decreased.

**Table 1 Buckling load maximization design**

| | Ply angles | | Buckling | Second |
|---|---|---|---|---|
| $N_y/N_x$ | $\theta_1$, deg | $\theta_2$, deg | load, GPa | eigenvalue |
| 0.25 | 52.8 | 52.6 | 293.9 | 294.1 |
| 0.50 | 62.1 | 59.6 | 221.3 | 221.4 |




$\bar{N}_y/\bar{N}_x = 0.25$




$\bar{N}_y/\bar{N}_x = 0.5$

**Fig. 2 Mode reliabilities of the deterministic designs in terms of correlation coefficient between the applied loads.**

The changes of mode reliabilities are different for each load case. In the case of $\bar{N}_y/\bar{N}_x = 0.25$, the first mode of $m + n = \text{odd}$ is dominant. As the correlation coefficient increases, the mode reliability of the first mode of $m + n = \text{even}$ approaches that of the first mode of $m + n = \text{odd}$. Then the two mode reliabilities take almost the identical value.

In the case of $\bar{N}_y/\bar{N}_x = 0.5$, the second mode of $m + n = \text{even}$ is dominant when the correlation coefficient is less than $-0.7$. However, when the correlation coefficient is greater than $-0.6$, mode reliabilities of both the first modes of $m + n = \text{even}$ and $m + n = \text{odd}$ take almost the same value.

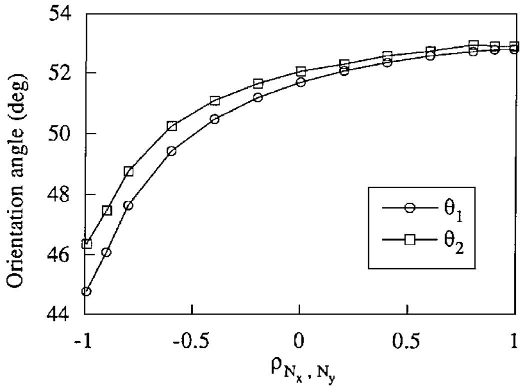
*Reliability-Based Optimization*

The optimum designs are investigated for the two load cases. The changes of optimum orientation angles in terms of the correlation coefficient $\rho_{N_x,N_y}$ are shown in Fig. 3. In both load cases, the optimum orientation angle approaches the deterministic design as the correlation coefficient becomes larger. In the case of $\bar{N}_y/\bar{N}_x = 0.5$, the reliability-based design is identical to the deterministic design when the correlation coefficient is larger than $-0.6$. This is because the deterministic design has two dominant mode reliabilities whose mode reliabilities are identical in the case of $\rho_{N_x,N_y} \ge -0.6$.

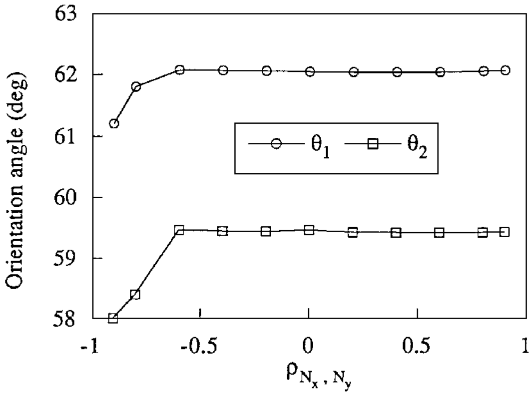
Variations of the applied loads yield changes in the resultant load directions, which have large effects on the reliability. However, when the correlation coefficient $\rho_{N_x,N_y}$ is large, the variations of the resultant load direction become small. Therefore, the reliability-based optimal design approaches the deterministic design.

### Correlation Between Ply Orientation Angles

Effects of correlation between ply orientation angles on the reliability and the reliability-based optimization are investigated. Only ply orientation angles of all layers are treated as dependent random




a) $\bar{N}_y/\bar{N}_x$ = 0.25




b) $\bar{N}_y/\bar{N}_x$ = 0.5

**Fig. 3 Optimum orientation angles of the reliability-based design subject to correlated loads.**

variables, whereas the material constants and the applied loads are set to be deterministic. Therefore, the number of random variables is eight.

The variances of orientation angles $\theta$ of all plies are assumed to take the same value, and their standard deviations (SDs) are set to $\mathrm{SD}(\theta) = 5$ deg. The correlation coefficients between layers are also assumed to take the same value. In this case, the correlation coefficient $\rho_\theta$ is bounded as $-1/7 \le \rho_\theta \le 1$.

*Reliability Analysis*

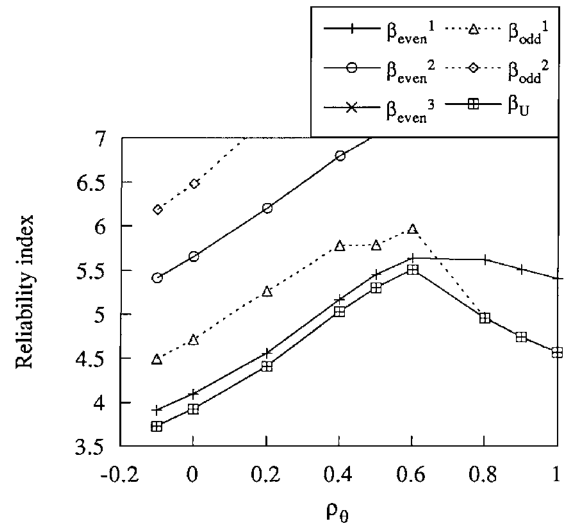
Reliability analyses are performed on the two deterministic designs: load cases of $N_y/N_x = 0.25$ and 0.5. The changes of the system reliability and the dominant mode reliabilities in terms of the correlation coefficient are shown in Fig. 4. The system reliabilities have a peak around the value 0.5–0.6 of the correlation coefficient $\rho_\theta$. This tendency is completely different from the preceding case, i.e., in case of correlation between the applied loads. This is because nonlinearity between buckling load and ply orientation angle is strong. Details are shown in the Appendix.

*Reliability-Based Optimization*

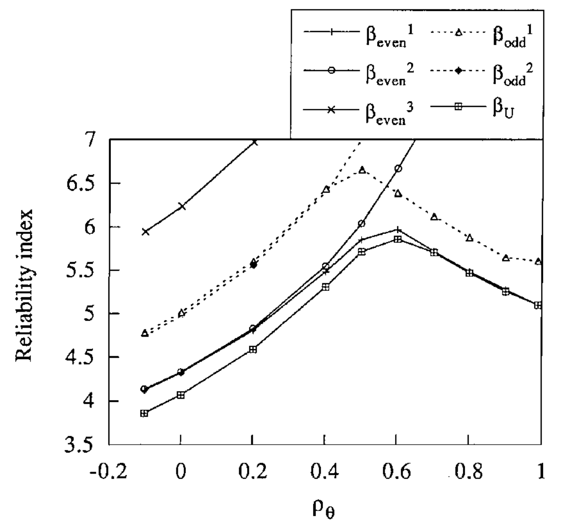
For the two load cases, the reliability-based designs are investigated. The changes of the system reliability and the dominant mode reliabilities in terms of the correlation coefficient are shown in Fig. 5. Each reliability-based design has two dominant failure modes, as in the other conditions.

Optimum orientation angles are shown in Fig. 6. In the case of $N_y/N_x = 0.25$, the optimum orientation angles are almost constant regardless of the values of the correlation coefficient. The ply angles do not approach the deterministic optimum design ($\theta_1 = 52.8$, $\theta_2 = 52.6$ deg) even if the correlation coefficient is increased. On the other hand, in the case of $N_y/N_x = 0.50$, the optimum orientation angles change drastically with respect to the correlation coefficient. Especially, in the region of $\rho_\theta \le 0.4$, the inner ply's angle $\theta_2$ is larger than the outer ply's angle $\theta_1$.

These results show that correlation between ply orientation angles plays an important role in the reliability-based design due to




$N_y/N_x$ = 0.25




$N_y/N_x$ = 0.5

**Fig. 4 Change in the reliability of the deterministic design in terms of correlation coefficient between ply orientation angles.**

the strong nonlinearity of buckling load in terms of the orientation angles.

*Effect of Correlation on Reliability-Based Design*

As shown in Fig. 6, the reliability-based design strongly depends on the correlation coefficient between ply orientation angles for the load case of $N_y/N_x = 0.50$. This result indicates the importance of correlation on the reliability.

To complement the discussion, the reliability-based design ignoring correlation is investigated to determine how the reliability is changed in terms of correlation. The reliability-based design, taking no account of correlation, gives the optimum ply orientation angles of (57.5, 78.4 deg) for the load case of $N_y/N_x = 0.50$ (Fig. 3b). The change of the reliability vs the correlation coefficient $\rho_\theta$ is illustrated in Fig. 7, where the corresponding curve is denoted Rel._based without $\rho$. In Fig. 7, the reliability of the design is compared with the reliability-based and the deterministic optimum designs, which are designated Rel._based with $\rho$ and Deterministic, respectively. The reliability of the reliability-based design disregarding correlation is much lower than a deterministic design for the case with a higher value of the correlation coefficient between the ply orientation angles, i.e., $\rho_\theta > 0.4$. This indicates that ignoring correlation yields a wrong result for the reliability-based design.

Note that the tendency depends on each load case. For the other load case of $N_y/N_x = 0.25$, the reliability-based optimum design does not depend much on the correlation coefficient. Nevertheless, it

**Table 2 Comparison of accuracy for correlated applied loads in the case of $\bar{N}_y/\bar{N}_x$ = 0.25**

| | Deterministic design | | | | Reliability-based design | | | |
|---|---|---|---|---|---|---|---|---|
| $\rho_{N_x,N_y}$ | $\hat{P}_f$ | COV, % | $\hat{\beta}$ | $\beta_U$ | $\hat{P}_f$ | COV, % | $\hat{\beta}$ | $\beta_U$ |
| −0.8 | 7.383E–15 | 9.04 | 7.690 | 7.703 | 1.176E–17 | 10.7 | 8.693 | 8.697 |
| −0.6 | 3.549E–13 | 7.97 | 7.178 | 7.177 | 2.036E–15 | 11.8 | 7.853 | 7.859 |
| 0.0 | 5.849E–10 | 9.15 | 6.084 | 6.074 | 1.797E–10 | 9.15 | 6.271 | 6.253 |
| 0.4 | 1.383E–8 | 7.52 | 5.556 | 5.564 | 1.200E–8 | 8.08 | 5.580 | 5.605 |
| 0.8 | 1.416E–7 | 5.03 | 5.134 | 5.145 | 1.282E–7 | 5.30 | 5.153 | 5.145 |

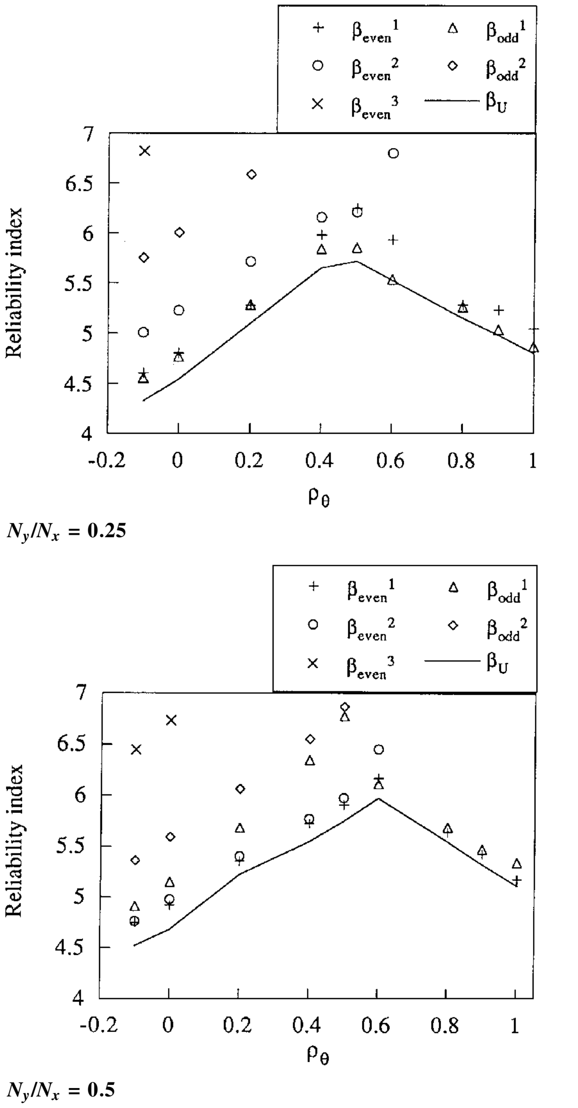



**Fig. 5 Change in the reliability of the reliability-based design in terms of correlation coefficient between ply orientation angles.**

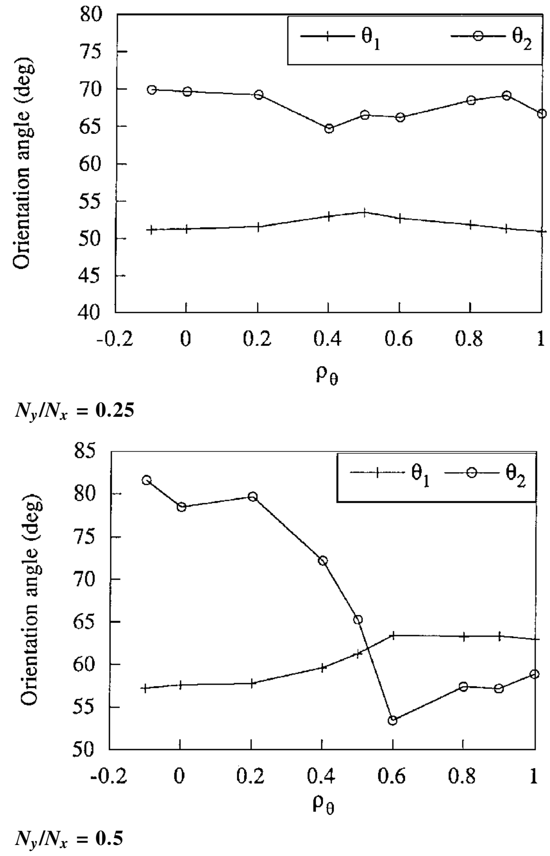



**Fig. 6 Optimum orientation angles of the reliability-based design in terms of correlation coefficient between ply orientation angles.**

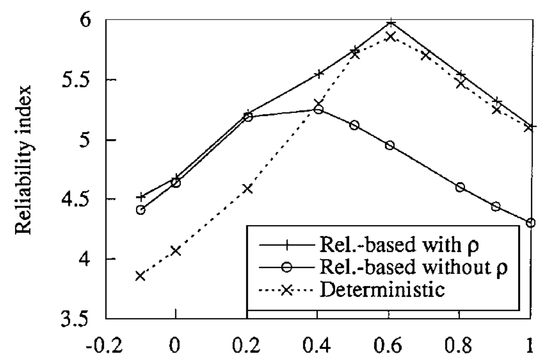



**Fig. 7 Comparison of the reliabilities for the designs with or without considering correlation under $N_y/N_x$ = 0.50.**

is important in the reliability-based design to quantitatively estimate the correlation coefficients between random variables.

**Comparison with Monte Carlo Simulation**

In the previous study,[10] the proposed FORM-based approach is confirmed to have sufficient accuracy for designing a laminated plate in comparison with a Monte Carlo simulation, although the accuracy is not sufficient to estimate the reliability quantitatively. Because the buckling load is a strongly nonlinear function of the ply orientation angles, the linearization of the buckling load using FORM yields an error in the estimated value of the reliability.

In this study, the accuracy is checked by performing a Monte Carlo simulation for the case of the correlated random variables. To reduce the computational costs, an efficient sampling method is adopted. The method is called the simulation outside the $\beta$ sphere.[21] In the method, no sample is generated in the region inside the distance $\beta$ from the origin of $\boldsymbol{U}$ space. The excluded region lies in the safety region if $\beta$ is less than or equal to the reliability index of the most dominant mode. Because the probability of the excluded region follows the chi-square distribution, the failure probability is calculated by a simple transformation. When the number of random variables is small, the method is very effective for reducing computational costs. The accuracy is confirmed for the cases where the applied loads and the orientation angles are random variables.

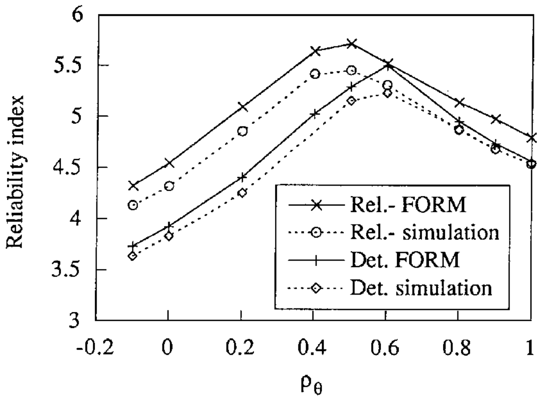



**Fig. 8 Comparison of FORM and Monte Carlo simulation in terms of correlation coefficient between ply orientation angles for the deterministic and the reliability-based designs with $N_y/N_x$ = 0.25.**

*Applied Load*

The reliability estimated by the Monte Carlo simulation is compared with that using FORM for the deterministic and the reliability-based designs under $\bar{N}_y/\bar{N}_x = 0.25$ in Table 2. The estimated failure probability $\hat{P}_f$ is calculated by performing $2 \times 10^6$ buckling analyses with the excluded region. The estimated COV of the failure probability is not small. However, it may be sufficient to compare the accuracy of FORM with that of the simulation.

The reliability index $\beta_U$ using FORM is found to take almost the same value as the reliability index $\hat{\beta}$ estimated by the simulation for all of the values of the correlation coefficient $\rho_{N_x,N_y}$ in both the deterministic and the reliability-based designs. This result indicates that FORM has enough accuracy. This is because the buckling load is a linear function in terms of the applied load as far as the identical mode is concerned. Hence, the reliability-based optimization by the FORM-based approach also has enough accuracy.

*Orientation Angle*

For the case where the ply orientation angles are correlated, the reliability using FORM is compared with that using the Monte Carlo simulation for several values of the correlation coefficient $\rho_\theta$.

The reliabilities for the deterministic and the reliability-based designs with $N_y/N_x = 0.25$ are compared in Fig. 8. The two designs are designated as Det. and Rel., respectively, whereas the terms FORM and simulation denote reliability indices evaluated by the FORM and the Monte Carlo simulation, respectively. In the simulation, the reliability is estimated by carrying out $2 \times 10^6$ buckling analyses with the excluded region as in the preceding case of the correlated applied loads. The estimated COV of the probability is less than 10%.

The results show that the reliability calculated by FORM is higher than that by the simulation for all of the values of the correlation coefficient $\rho_\theta$ in both the deterministic and the reliability-based designs. This is because the buckling load is strongly nonlinear in terms of the ply orientation angles. However, the difference of the estimated reliability in terms of the correlation coefficient is very similar for both methods. Comparing the reliabilities estimated by the simulation method between the reliability-based and the deterministic designs, the reliability-based design using the FORM-based approach has higher reliability than the deterministic design for all of the values of the correlation coefficient. The difference becomes small for $\rho_\theta > 0.8$. This result indicates that the FORM-based approach is effective for the reliability-based optimization of the laminated plate subject to buckling when the ply orientation angles are the correlated random variables.

## Concluding Remarks

The reliability-based design approach is applied to the optimum design of composite laminated plates subject to buckling. The reliability is evaluated by modeling the plate as a series system consisting of significant eigenmodes. To evaluate the mode reliability for an intended mode, mode tracking strategy is used.

The effects of correlation between random variables on the reliability and the reliability-based design for laminated composite plates subject to buckling are investigated. In the numerical calculations, two cases of the random variables are considered. The first is correlation between the applied loads, and the second is correlation between the orientation angles. In the first case, the reliability is decreased as the correlation coefficients are increased. Moreover, the reliability-based design approaches the deterministic design as the correlation coefficient between applied loads is increased. This is because the variation of the resultant load direction is decreased. On the other hand, the reliability-based design is different from the deterministic design when the correlation coefficient between the loads is small, that is, the variation of the resultant load direction is large. Then it is shown that the effect of correlation between ply orientation angles is different from the preceding two cases. Because the buckling load is strongly nonlinear in terms of the ply orientation angles, the reliability has a peak at some positive value of the correlation coefficient. The reliability-based design does not reach the deterministic optimum design when the value of the correlation coefficient is increased. These results show that correlation between ply orientation angles plays an important role in the reliability analysis and the reliability-based design of fiber-reinforced laminates subject to buckling.

Finally, the reliability evaluated by the FORM-based approach is compared with the reliability by Monte Carlo simulation. The result shows that the reliability indices estimated by both methods are almost identical and that the reliability-based design using the FORM-based approach has higher reliability than a deterministic optimum design. Therefore, the FORM-based approach is effective for the reliability-based optimization of a laminated plate subject to buckling when the variables are correlated.

## Appendix: Effect of Nonlinearity of Limit State Function on Reliability

When correlations between the ply orientation angles exist, the system reliabilities have a peak when the correlation coefficient $\rho_\theta$ is around 0.4–0.6, as shown in Figs. 4 and 5. This is because the buckling load is strongly nonlinear in terms of ply orientation angles.

Here, it is shown how the strong nonlinear limit state function affects nonmonotonical changes of the reliability. Consider a simple two-dimensional, normally distributed random vector $\boldsymbol{X}$. The mean, the SD, and the correlation coefficient are described as $\bar{\boldsymbol{X}} = (\bar{X}_1, \bar{X}_2)$, $(\sigma_{X_1}, \sigma_{X_2})$, and $\rho$, respectively. For simplicity, the variations are assumed to take the same value, $(\sigma = \sigma_{X_1} = \sigma_{X_2})$. The probability density contour curve of the random vector is expressed by an ellipse. The center is located in the mean value point $\bar{\boldsymbol{X}}$, and the principal axis declines 45 deg in the $\boldsymbol{X}$ space, as shown in Fig. A1. The major axis declines +45 deg if $\rho$ is positive. On the other hand, the major axis declines −45 deg if $\rho$ is negative. The eccentricity is $\sqrt{[2\rho^2/(1+\rho^2)]}$. As the absolute value of the correlation coefficient is larger, the ellipse is thinner.

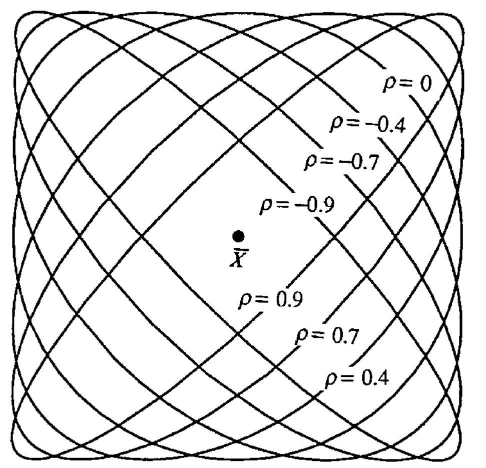



**Fig. A1 Probability density contours for correlated normal random vector.**

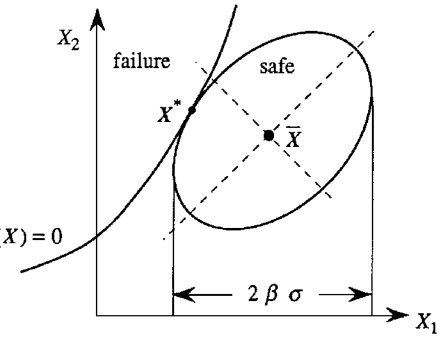



**Fig. A2 Design point in *X* space for the correlated random variables.**

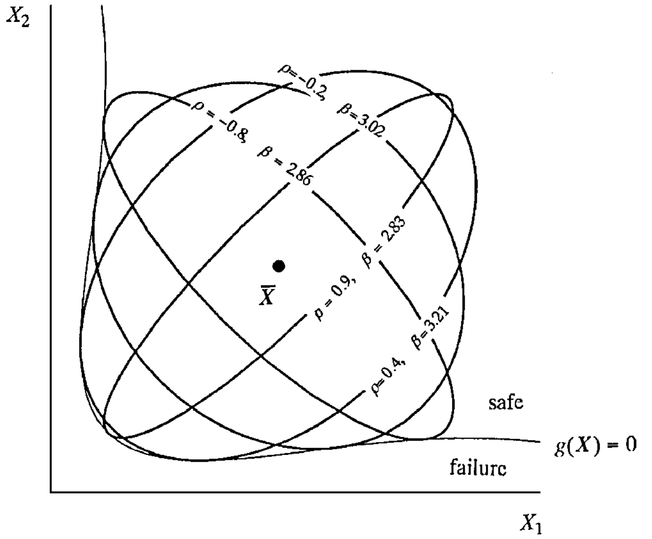



**Fig. A3 Change of reliability index with respect to correlation coefficient for a nonlinear limit state function.**

The design point in the $X$ space is located in a point of a contact between the limit state curve and the ellipse, as shown in Fig. A2. The length of the ellipse projected on the coordinate axis is described as $2\beta\sigma$, where $\beta$ is a reliability index. The design point and the reliability index differ as the correlation coefficient changes.

Figure A3 shows the reliability index for various values of the correlation coefficient, where a limit state function is strongly nonlinear in the $X$ space. The ellipses are the probability density contours osculating the limit state curve. The reliability index is found to have a peak around $\rho = 0.4$.

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